

On the coloring of 3-element subsets*

D. Zakharov

Theorem 1 *Let $p = 8k - 1$ be a prime. Then we can color the set $\binom{R_{p+2}}{3}$ of 3-element subsets of $R_{p+2} = \{1, \dots, p+2\}$ into p colors such that any two sets $x, y \in \binom{R_{p+2}}{3}$, $|x \cap y| = 2$ have different colors.*

Professor Raigorodskiy confirmed that this result is new.

Remarks. (a) Obviously, the set $\binom{R_{p+2}}{3}$ can not be colored in a fewer than p colors because in the set

$$\{\{1, 2, 3\}, \{1, 2, 4\}, \dots, \{1, 2, p+2\}\}$$

any pair of triplets must have different colors.

(b) It is clear from the proof that the theorem also holds for primes p such that $-1 \not\equiv 2^r \pmod{p}$ for each r .

(c) We can consider graph $G(n, 3, 2)$ (see [1-6]) which has vertices set $\binom{R_n}{3}$ that are connected if their intersection has two elements, i.e.

$$G(n, 3, 2) = (V, E), \quad V = \binom{R_n}{3}, \quad E = \{\{x, y\} : |x \cap y| = 2\}$$

In this terminology the theorem states that $\chi(G(p+2, 3, 2)) \leq p$ (and so $\chi(G(p+2, 3, 2)) = p$).

Acknowledgements. I would like to acknowledge my advisor prof. A. M. Raigorodskiy for his attention to this work.

1 The proof of theorem 1

1.1 Construction of coloring

Let $V = \binom{R_n}{3}$, $N = \{p+1, p+2\}$. Consider 4 sets:

$$V_0 = \{v \in V, v \cap N = \emptyset\},$$

$$V_2 = \{v \in V, N \subset v\},$$

$$W_1 = \{v \in V, v \cap N = \{p+1\}\},$$

$$W_2 = \{v \in V, v \cap N = \{p+2\}\}$$

Every vertex $x = \{x_1, x_2, x_3\} \in V_0$ we paint in color

$$c(x) = x_1 + x_2 + x_3 \pmod{p},$$

vertex $x = (x_1, p+2, p+1) \in V_2$ we paint in color

$$c(x) = 3x_1 \pmod{p}.$$

For the coloring of W_1 and W_2 we will use the following lemma, which will be proved in 2.3.

Let $\tilde{R}_p = \{(x, y) | x, y \in R_p, x \neq y\}$.

*This paper is prepared under the supervision of A.M. Raigorodskiy and is submitted to the Moscow Mathematical Conference for High-School Students. Readers are invited to send their remarks and reports on this paper to mmks@mccme.ru

Lemma 1 *There exists a map $f : \widetilde{R_p} \rightarrow \mathbb{Z}_2$ such that*

1. $f(x, y) \neq f(y, x)$ for each x, y
2. $f(x, y) \neq f(\frac{x+y}{2}, x)$ for each x, y (here the division is the division in \mathbb{Z}_p),

Define a function $f_1 : \widetilde{R_p} \rightarrow \mathbb{Z}_p$ by $f_1(x, y) = x$ if $f(x, y) = 0$ and $f_1(x, y) = y$ if $f(x, y) = 1$. Define a function $f_2 : \widetilde{R_p} \rightarrow \mathbb{Z}_p$ by $f_2(x, y) = x + y - f_1(x, y)$. Now let us paint vertex

$$x = (x_1, x_2, p + i) \in W_i$$

in color

$$c(x) = x_1 + x_2 + f_i(x_1, x_2) \pmod{p}.$$

Obviuosly, we have constructed a coloring of all elements of $\binom{R_{p+2}}{3}$ in p colors.

1.2 The proof that coloring is regular

Let us take any two elements

$$x = (x_1, x_2, x_3), \quad y = (x_1, x_2, x_4) \in \binom{R_{p+2}}{3}$$

and consider following cases:

Case 1: $x, y \in V_0$. Then, obviously,

$$c(x) \equiv x_1 + x_2 + x_3 \not\equiv x_1 + x_2 + x_4 \equiv c(y) \pmod{p}.$$

Case 2: $x, y \in V_2$, $x_1, x_2 \in N$. Then

$$c(x) \equiv 3x_3 \not\equiv 3x_4 \equiv c(y) \pmod{p},$$

because $p > 3$ is prime, i.e. p is not divisible by 3.

Case 3: $x \in V_0$, $y \in V_2$. This case is impossible.

Case 4: $x \in W_i$, $y \in V_2$. We can assume that $x_1 = p + 2$, $x_4 = p + 1$, so

$$c(x) - c(y) \equiv x_2 + x_3 + f_i(x_2, x_3) - 3x_2 \equiv x_3 + f_i(x_2, x_3) - 2x_2 \pmod{p}.$$

If $f_i(x_2, x_3) = x_2$, then

$$c(x) - c(y) \equiv x_3 - x_2 \not\equiv 0 \pmod{p},$$

else $f_i(x_2, x_3) = x_3$ and

$$c(x) - c(y) \equiv 2x_3 - 2x_2 \not\equiv 0 \pmod{p},$$

Case 5: $x \in W_i$, $y \in V_0$, $x_3 \in N$. We have

$$c(x) - c(y) \equiv x_1 + x_2 + f_i(x_1, x_2) - x_1 - x_2 - x_4 \equiv f_i(x_1, x_2) - x_4 \not\equiv 0 \pmod{p}$$

by definition of f_i .

Case 6: $x \in W_1, y \in W_2$. In this case $x_3 = p + 1, x_4 = p + 2$, consequently

$$c(x) - c(y) \equiv f_1(x_1, x_2) - f_2(x_1, x_2) \not\equiv 0 \pmod{p}$$

by definition of f_1 and f_2 .

Case 7: $x, y \in W_1, x_1 = p + 1$. We can write that

$$c(x) - c(y) \equiv x_2 + x_3 + f_1(x_2, x_3) - x_2 - x_4 - f_1(x_2, x_4) \equiv x_3 + f_1(x_2, x_3) - x_4 - f_1(x_2, x_4) \pmod{p}.$$

Consider subcases.

Subcase 7.1: $f_1(x_2, x_3) = f_1(x_2, x_4) = x_2$. Then

$$x_3 + f_1(x_2, x_3) - x_4 - f_1(x_2, x_4) \equiv x_3 - x_4 \not\equiv 0 \pmod{p}.$$

Subcase 7.2: $f_1(x_2, x_3) = x_3, f_1(x_2, x_4) = x_4$. Then

$$x_3 + f_1(x_2, x_3) - x_4 - f_1(x_2, x_4) \equiv 2x_3 - 2x_4 \not\equiv 0 \pmod{p}.$$

Subcase 7.3: $f_1(x_2, x_3) = x_2, f_1(x_2, x_4) = x_4$. Then

$$x_3 + f_1(x_2, x_3) - x_4 - f_1(x_2, x_4) \equiv x_3 + x_2 - 2x_4 \pmod{p}.$$

Suppose $x_3 + x_2 \equiv 2x_4 \pmod{p}$. So we have $x_4 = \frac{x_2 + x_3}{2}$. From this we get

$$f_1(x_2, x_3) = x_2, \quad f_1\left(x_2, \frac{x_2 + x_3}{2}\right) = \frac{x_2 + x_3}{2}.$$

And so

$$f(x_2, x_3) = 1, \quad f\left(x_2, \frac{x_2 + x_3}{2}\right) = 0 \Rightarrow f(x_3, x_2) = 0$$

which contradicts to the properties of f . Hence, $c(x) - c(y) \not\equiv 0 \pmod{p}$.

Thus, we considered all cases and the constructed coloring is regular.

1.3 The proof of lemma 1

Lemma 2 Let $p = 8k - 1$ be a prime. Then $-1 \not\equiv 2^r \pmod{p}$ for each r .

Proof. Denote by d the order of 2 modulo p . We denote by $\left(\frac{a}{p}\right)$ the Legendre symbol. It's known, that

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = (-1)^{\frac{64k^2-16k+1-1}{8}} = 1.$$

Thus, 2 is a quadratic residue in \mathbb{Z}_p . Then $2^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ and so $d \mid \frac{p-1}{2} = 4k - 1$, i.e. d is odd.

Suppose, that there exists minimal number l , such that $2^l \equiv -1 \pmod{p}$. Then $d \mid 2l$ but d is odd, whence $d \mid l$ and $2^l \equiv 1 \pmod{p}$. Lemma 2 is proved.

Let us define a graph G with vertex set $\widetilde{R_p}$ and in which we connect (x, y) with

$$(y, x), \left(\frac{x+y}{2}, x\right), (y, 2x-y)$$

It is easy to see that if (x, y) and (a, b) are connected then $\frac{x-y}{a-b} = -2^s$ for some s . So if there is an odd cycle $(x_1, y_1), \dots, (x_l, y_l)$ then

$$x_1 - y_1 \equiv -2^{s_2}(x_2 - y_2) \equiv \dots \equiv (-1)^{l-1}2^{s_2+\dots+s_l}(x_l - y_l) \equiv (-1)^l2^{s_1+\dots+s_l}(x_1 - y_1)$$

And we get

$$(x_1 - y_1)(2^S + 1) \equiv 0 \pmod{p}$$

which contradicts to Lemma 2. Then G has not odd cycles so it is bipartite.

Now let us take some 2-coloring of G $V = \mathcal{M}_1 \cup \mathcal{M}_2$. We define the map f as:

$$f(x, y) = 1 \Leftrightarrow (x, y) \in \mathcal{M}_1$$

Clearly $f(x, y) \neq f(y, x)$ and $f(x, y) \neq f(\frac{x+y}{2}, x)$ because corresponding vertices are adjacent. Lemma 1 is proved.

References

- [1] A.M. Raigorodskii, *Cliques and cycles in distance graphs and graphs of diameters*, “Discrete Geometry and Algebraic Combinatorics”, AMS, Contemporary Mathematics, 625 (2014), 93 - 109.
- [2] A.M. Raigorodskii, *Coloring Distance Graphs and Graphs of Diameters*, Thirty Essays on Geometric Graph Theory, J. Pach ed., Springer, 2013, 429 - 460.
- [3] B. Bollobás, B.P. Narayanan, A.M. Raigorodskii, *On the stability of the Erdős–Ko–Rado theorem*, J. Comb. Th. Ser. A, 137 (2016), 64 - 78.
- [4] A.M. Raigorodskii, *Combinatorial geometry and coding theory*, Fundamenta Informatica, 145 (2016), 359 - 369.
- [5] A.V. Bobu, O.A. Kostina, A.E. Kupriyanov, *Independence numbers and chromatic numbers of some distance graphs*, Problemy Peredachi Informatsii, 2015, Vol. 51, No. 2, pp. 86–98.
- [6] J. Balogh, A.V. Kostochka, A.M. Raigorodskii, *Coloring some finite sets in \mathbb{R}^n* , Discussiones Mathematicae Graph Theory, 33 (2013), N1, 25 - 31.